SOME LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTIONS RELATED TO THE GAMMA FUNCTION

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Abstract. In this article, logarithmically complete monotonicity properties of some functions such as $\frac{1}{[\Gamma(x+1)]^{1/x}}$, $\frac{[\Gamma(x+\alpha+1)]^{1/(x+\alpha)}}{[\Gamma(x+1)]^{1/x}}$, $\frac{[\Gamma(x+1)]^{1/x}}{(x+1)^\alpha}$ and $\frac{[\Gamma(x+1)]^{1/x}}{x^\alpha}$ defined in $(-1,\infty)$ or $(0,\infty)$ for given real number $\alpha\in\mathbb{R}$ are obtained, some known results are recovered, extended and generalized. Moreover, some basic properties of the logarithmically completely monotonic functions are established.

1. Introduction

Recall [32, Chapter XIII] and [66, Chapter IV] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^k f^{(k)}(x)\ge 0 \tag{1}$$

for all $k\ge 0$ on I. For our own convenience, let $\mathcal{C}[I]$ denote the set of completely monotonic functions on I. The well-known Bernstein's Theorem in [66, p. 160, Theorem 12a] states that a function f on $[0,\infty)$ is completely monotonic if and only if there exists a bounded and non-decreasing function $\alpha(t)$ such that

$$f(x)=\int_0^\infty e^{-xt}\,\mathrm{d}\,\alpha(t) \tag{2}$$

converges for $x\in[0,\infty)$. This tells us that $f\in\mathcal{C}[[0,\infty)]$ if and only if it is a Laplace transform of the measure α. There have been a lot of literature about the completely monotonic functions, for examples, [3, 4, 5, 22, 25, 32, 34, 35, 39, 40, 42, 58, 60, 61, 62, 63, 66] and references therein.

Recall also [6, 43, 47] that a positive function f is said to be logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n[\ln f(x)]^{(n)}\ge 0 \tag{3}$$

for all $x\in I$ and $n\in\mathbb{N}$. For simplicity, let $\mathcal{L}[I]$ stand for the set of logarithmically completely monotonic functions on I.

Among other things, it is proved in [7, 10, 43, 47, 56] that a logarithmically completely monotonic function is always completely monotonic, that is, $\mathcal{L}[I]\subset\mathcal{C}[I]$, but not conversely, since a convex function may not be logarithmically convex (see [33, p. 7, Remark. 1.16]).

Recall [66] that a function f defined in $(0,\infty)$ is called a Stieltjes transform if it can be of the form

$$f(x)=a+\int_0^\infty \frac{1}{s+x}\mathrm{d}\,\mu(s), \tag{4}$$

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where a is a nonnegative number and μ a nonnegative measure on $[0,\infty)$ satisfying

$$\int_0^\infty \frac{1}{1+s}\,\mathrm{d}\mu(s) < \infty. \tag{5}$$

The set of Stieltjes transforms is denoted by $\mathcal{S}$.

Motivated by the papers [47, 54], among other things, it is further revealed in [7] that

$$\mathcal{S}\setminus\{0\} \subset \mathcal{L}[(0,\infty)] \subset \mathcal{C}[(0,\infty)]. \tag{6}$$

In [7, Theorem 1.1] and [22, 53] it is pointed out that logarithmically completely monotonic functions on $(0,\infty)$ can be characterized as the infinitely divisible completely monotonic functions studied by Horn in [24, Theorem 4.4]. The functions in $\mathcal{L}[I]$ are also characterized by $-\frac{f'}{f} \in \mathcal{C}[I]$. Recently it is found that a finer inclusion

$$\mathcal{S} \subset \mathcal{C}^*[(0,\infty)] \subset \mathcal{L}[(0,\infty)] \subset \mathcal{C}[(0,\infty)] \tag{7}$$

had been established in [8, Section 14.2, pp. 122–127] and [61], where $\mathcal{C}^*[(0,\infty)]$ denotes the set

$$\left\{ f(x) \,\middle|\, \left[\frac{1}{f(x)}\right]' \in \mathcal{C}[(0,\infty)] \right\}. \tag{8}$$

In [8, p. 122], it was proved that

$$\mathcal{H}\setminus\{0\} \subset \mathcal{P} = \mathcal{C}^*[(0,\infty)] \tag{9}$$

and it was told that this is a theorem of F. Hirsch with due reference. This result says that if $f>0$ and $f\in\mathcal{H}$ then $\frac{1}{f}$ is the Laplace transform of a potential kernel, hence the Laplace transform of an infinitely divisible measure and f is a Bernstein function, i.e. a positive function whose derivative is completely monotonic. On [8, p. 127] it is proved that $\mathcal{S}\subset\mathcal{H}$.

From Bernstein's Theorem it also follows that completely monotonic functions on $(0,\infty)$ are always strictly completely monotonic unless they are constant, see [19, 53] and [61, p. 11]. Also it follows that a logarithmically completely monotonic function on $(0,\infty)$ is strictly so unless it is of the form $c\exp(-\alpha x)$ for $c>0$ and $\alpha\ge 0$, so there is no need to discuss the sharpening with "strictly" in general. If its representing measure of a function f as a Stieltjes transform is concentrated on $[a,\infty)$ with $a>0$, then $f\in\mathcal{L}[(-a,\infty)]$.

The classical Euler gamma function is usually defined for $\operatorname{Re} z>0$ by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t. \tag{10}$$

The logarithmic derivative of the gamma function

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \tag{11}$$

is called the psi or digamma function and $\psi^{(n)}(x)$ for $n\in\mathbb{N}$ the polygamma functions. It is well-known that the gamma function is a very important classical special function and has many applications [1, 10, 20, 29]. One of the reasons why the gamma function is still interesting, although nearly three centuries have elapsed after its first appearance, is that it has many applications to various areas of mathematics ranging from probability theory to number theory and function theory. (Logarithmically) completely monotonic functions have applications in many branches. For example, they play a role in complex analysis, number theory, potential theory, probability theory [10], physics [29], numerical and asymptotic analysis, integral transforms [66], and combinatorics. Some related references are listed in [3, 4, 5, 7, 22, 32, 48, 49, 53, 66]. In recent years, inequalities and (logarithmically) completely monotonic functions involving the

gamma, psi, or polygamma functions are established by some mathematicians (see [2, 3, 4, 5, 12, 13, 14, 15, 16, 17, 18, 22, 23, 26, 30, 44, 45, 46, 50, 51, 52, 63] and the references therein).

In this paper, using Leibniz's Identity, the discrete and integral representations of polygamma functions and other analytic techniques, some functions such as

$$\frac{1}{[\Gamma(x+1)]^{1/x}},\quad \frac{[\Gamma(x+\alpha+1)]^{1/(x+\alpha)}}{[\Gamma(x+1)]^{1/x}},\quad \frac{[\Gamma(x+1)]^{1/x}}{(x+1)^\alpha}\quad \text{and}\quad \frac{[\Gamma(x+1)]^{1/x}}{x^\alpha} \tag{12}$$

with $x\in(-1,\infty)$ or $x\in(0,\infty)$ for given real number $\alpha\in\mathbb{R}$ are shown to be logarithmically completely monotonic. Moreover, some basic properties of the logarithmically completely monotonic functions are established.

Our main results are as follows.

Theorem 1.

$$\frac{1}{[\Gamma(x+1)]^{1/x}}\in\mathcal{L}[(-1,\infty)]. \tag{13}$$

Theorem 2.

$$\frac{[\Gamma(x+1)]^{1/x}}{(x+1)^\alpha}\in\mathcal{L}[(-1,\infty)] \tag{14}$$

if and only if $\alpha\ge 1$.

Theorem 3.

$$\frac{x^\alpha}{[\Gamma(x+1)]^{1/x}}\in\mathcal{L}[(0,\infty)] \tag{15}$$

if and only if $\alpha\le 0$.

$$\frac{[\Gamma(x+1)]^{1/x}}{x^\alpha}\in\mathcal{L}[(0,\infty)] \tag{16}$$

if and only if $\alpha\ge 1$.

Theorem 4. *Let*

$$\tau(s,t)=\frac{1}{s}\left[t-(t+s+1)\left(\frac{t}{t+1}\right)^{s+1}\right] \tag{17}$$

for $(s,t)\in(0,\infty)\times(0,\infty)$ *and* $\tau_0=\tau(s_0,t_0)>0$ *be the maximum of* $\tau(s,t)$ *on the set* $\mathbb{N}\times(0,\infty)$. *Then for any given real number* α *satisfying* $\alpha\le\frac{1}{1+\tau_0}<1$,

$$\frac{(x+1)^\alpha}{[\Gamma(x+1)]^{1/x}}\in\mathcal{L}[(-1,\infty)]. \tag{18}$$

Theorem 5. *For* $\alpha\le 0$ *such that* x^α *is real in* $(-1,0)$,

$$\frac{[\Gamma(x+1)]^{1/x}}{x^\alpha}\in\mathcal{L}[(-1,0)]. \tag{19}$$

For $\alpha\ge 1$ *such that* x^α *is real in* $(-1,0)$,

$$\frac{x^\alpha}{[\Gamma(x+1)]^{1/x}}\in\mathcal{L}[(-1,0)]. \tag{20}$$

As basic properties of the logarithmically completely monotonic functions, we obtain the following theorems.

Theorem 6. *Let* $f(x)\in\mathcal{L}[I]$. *Then* $\frac{f(x)}{f(x+\alpha)}\in\mathcal{L}[J]$ *if and only if* $\alpha>0$, *where* $J=I\cap\{x+\alpha\in I\}$.

Theorem 7. *Let* $f_i(x)\in\mathcal{L}[I]$ *and* $\alpha_i\ge 0$ *for* $1\le i\le n$ *with* $n\in\mathbb{N}$. *Then*

$$\prod_{i=1}^{n}[f_i(x)]^{\alpha_i}\in\mathcal{L}[I]. \tag{21}$$

Theorem 8. *Let $h'(x) \in \mathcal{C}[I]$ and $f(x) \in \mathcal{L}[h(I)]$. Then $f \circ h(x) = f\big(h(x)\big) \in \mathcal{L}[I]$.*

In Section 2, we are about to give proofs of these theorems. In Section 3, some remarks are given, some new results are deduced, and some known results are recovered, as applications of these theorems.

2. Proofs of theorems

It is well-known (see [1, 20, 64, 65] and [29, p. 16]) that the polygamma functions $\psi^{(k)}(x)$ can be expressed for $x > 0$ and $k \in \mathbb{N}$ as

$$\psi^{(k)}(x) = (-1)^{k+1} k! \sum_{i=0}^{\infty} \frac{1}{(x+i)^{k+1}} \tag{22}$$

or

$$\psi^{(k)}(x) = (-1)^{k+1} \int_0^{\infty} \frac{t^k e^{-xt}}{1-e^{-t}} \,\mathrm{d}\, t. \tag{23}$$

The first proof of Theorem 1. Let

$$g(x) = \begin{cases} \dfrac{\ln \Gamma(x+1)}{x}, & x \neq 0 \\ -\gamma, & x = 0 \end{cases} \tag{24}$$

for $x \in (-1, \infty)$, where $\gamma = 0.57721566\cdots$ is the Euler-Mascheroni constant. By direct calculation and using Leibniz's Identity, we obtain for $n \in \mathbb{N}$,

$$g^{(n)}(x) = \begin{cases} \dfrac{1}{x^{n+1}} \displaystyle\sum_{k=0}^{n} \frac{(-1)^{n-k} n! x^k \psi^{(k-1)}(x+1)}{k!} \triangleq \frac{h_n(x)}{x^{n+1}}, & x \neq 0, \\ \dfrac{\psi^{(n)}(1)}{n+1}, & x = 0, \end{cases} \tag{25}$$

$$h_n'(x) = x^n \psi^{(n)}(x+1) \begin{cases} > 0 & \text{in } (0,\infty) \text{ if } n \text{ is odd}, \\ \leq 0 & \text{in } (-1,0] \text{ if } n \text{ is odd}, \\ \leq 0 & \text{in } (-1,\infty) \text{ if } n \text{ is even}, \end{cases} \tag{26}$$

where

$$\psi^{(-1)}(x+1) = \ln \Gamma(x+1) \quad \text{and} \quad \psi^{(0)}(x+1) = \psi(x+1).$$

Hence, if n is odd the function $h_n(x)$ increases in $(0,\infty)$ and decreases in $(-1,0)$, if n is even it decreases in $(-1,\infty)$. Since $h_n(0) = 0$, it is easy to see that $h_n(x) \geq 0$ in $(-1,\infty)$ if n is odd and that $h_n(x) \geq 0$ in $(-1,0)$ and $h_n(x) \leq 0$ in $(0,\infty)$ if n is even. Thus, in the interval $(-1,\infty)$, the function $g^{(n)}(x) \geq 0$ if n is odd and $g^{(n)}(x) \leq 0$ if n is even. Since

$$\lim_{x\to\infty} \frac{\psi^{(k)}(x+1)}{x^{n-k}} = 0$$

for $-1 \leq k \leq n-1$, it follows that $\lim_{x\to\infty} g^{(n)}(x) = 0$. Consequently,

$$(-1)^{n+1} g^{(n)}(x) > 0$$

in $(-1,\infty)$ for $n \in \mathbb{N}$. This implies that

$$(-1)^k \{\ln[\Gamma(x+1)]^{1/x}\}^{(k)} < 0$$

in $(-1,\infty)$ for $k \in \mathbb{N}$ and the function $\frac{1}{[\Gamma(x+1)]^{1/x}}$ is logarithmically completely monotonic in $(-1,\infty)$. $\square$

The second proof of Theorem 1. It is not difficult to see that

$$g(x)=\frac{\ln\Gamma(x+1)-\ln\Gamma(1)}{x}=\frac{1}{x}\int_0^x\psi(t+1)\,\mathrm{d}\,t=\int_0^1\psi(xs+1)\,\mathrm{d}\,s \tag{27}$$

and

$$g^{(n)}(x)=\int_0^1 s^n\psi^{(n)}(xs+1)\,\mathrm{d}\,s. \tag{28}$$

Thus, the required result follows from using formula (22) or (23) in (28). □

Proof of Theorem 2. Let

$$\nu_\alpha(x)=\begin{cases}\dfrac{[\Gamma(x+1)]^{1/x}}{(x+1)^\alpha}, & x\ne 0\\ e^{-\gamma}, & x=0\end{cases} \tag{29}$$

for $x\in(-1,\infty)$. Then for $n\in\mathbb{N}$, by using (22),

$$\ln\nu_\alpha(x)=\frac{\ln\Gamma(x+1)}{x}-\alpha\ln(x+1), \tag{30}$$

$$[\ln\nu_\alpha(x)]^{(n)}=\frac{1}{x^{n+1}}\left[h_n(x)+\frac{(-1)^n(n-1)!\alpha x^{n+1}}{(x+1)^n}\right]\triangleq\frac{\mu_{\alpha,n}(x)}{x^{n+1}}, \tag{31}$$

$$\begin{aligned}\mu_{\alpha,n}'(x)&=x^n\psi^{(n)}(x+1)+\frac{(-1)^n(n-1)!\alpha x^n(x+n+1)}{(x+1)^{n+1}}\\&=x^n\left[\psi^{(n)}(x+1)+\frac{(-1)^n(n-1)!\alpha}{(x+1)^n}+\frac{(-1)^n n!\alpha}{(x+1)^{n+1}}\right]\\&=x^n\Bigg\{(-1)^{n+1}n!\sum_{i=1}^\infty\frac{1}{(x+i)^{n+1}}\\&\quad+(-1)^n(n-1)!\alpha\sum_{i=1}^\infty\left[\frac{1}{(x+i)^n}-\frac{1}{(x+i+1)^n}\right]\\&\quad+(-1)^n n!\alpha\sum_{i=1}^\infty\left[\frac{1}{(x+i)^{n+1}}-\frac{1}{(x+i+1)^{n+1}}\right]\Bigg\}\\&=(-1)^n(n-1)!x^n\sum_{i=1}^\infty\left[\frac{\alpha}{(x+i)^n}-\frac{\alpha}{(x+i+1)^n}\right.\\&\quad\left.-\frac{n\alpha}{(x+i+1)^{n+1}}+\frac{n(\alpha-1)}{(x+i)^{n+1}}\right]\\&=(n-1)!(-x)^n\sum_{i=1}^\infty\frac{[\alpha y+n(\alpha-1)](y+1)^{n+1}-\alpha(y+n+1)y^{n+1}}{y^{n+1}(y+1)^{n+1}}\\&=(n-1)!(-x)^n\sum_{i=1}^\infty\frac{\alpha[(y+n)(y+1)^{n+1}-(y+n+1)y^{n+1}]-n(y+1)^{n+1}}{y^{n+1}(y+1)^{n+1}}\\&=n!(-x)^n\sum_{i=1}^\infty\frac{1}{y^{n+1}}\left\{\alpha\left[1+\frac{1}{n}\left\langle y-(y+n+1)\left(\frac{y}{y+1}\right)^{n+1}\right\rangle\right]-1\right\},\end{aligned} \tag{32}$$

where $y=x+i>0$.

In [11, p. 28], [27, p. 154] and [28], Bernoulli's inequality states that if $x\ge-1$ and $x\ne0$ and if $\alpha>1$ or if $\alpha<0$ then

$$(1+x)^\alpha>1+\alpha x.$$

This means that
$$1+\frac{s+1}{t}<\left(1+\frac{1}{t}\right)^{s+1}$$
which is equivalent to
$$t-(t+s+1)\left(\frac{t}{t+1}\right)^{s+1}>0$$
for $s>0$ and $t>0$, then the function $\tau(s,t)$ defined by (17) is positive for $(s,t)\in(0,\infty)\times(0,\infty)$.

Since $\tau(s,t)>0$, it is deduced that
$$[\alpha y+n(\alpha-1)](y+1)^{n+1}-\alpha(y+n+1)y^{n+1}>0$$
for $y=x+i>0$ and $n\in\mathbb{N}$ if $\alpha\ge1$. This means that for $\alpha\ge1$,
$$\mu_{\alpha,n}'(x)\begin{cases}>0 & \text{in } (-1,0)\cup(0,\infty) \text{ if } n \text{ is even,}\\ >0 & \text{in } (-1,0) \text{ if } n \text{ is odd,}\\ <0 & \text{in } (0,\infty) \text{ if } n \text{ is odd,}\end{cases}$$
hence, it is obtained that the function $\mu_{\alpha,n}(x)$ is strictly increasing in $(-1,\infty)$ if n is even and that the function $\mu_{\alpha,n}(x)$ is strictly increasing in $(-1,0)$ and strictly decreasing in $(0,\infty)$ if n is odd. Since $\mu_{\alpha,n}(0)=0$, it follows that $\mu_{\alpha,n}(x)\le0$ in $(-1,\infty)$ if n is odd and that $\mu_{\alpha,n}(x)\le0$ in $(-1,0)$ and $\mu_{\alpha,n}(x)>0$ in $(0,\infty)$ if n is even. From $\lim_{x\to\infty}[\ln\nu_\alpha(x)]^{(n)}=0$, it is concluded that $[\ln\nu_\alpha(x)]^{(n)}\ge0$ in $(-1,\infty)$ if n is even and $[\ln\nu_\alpha(x)]^{(n)}\le0$ in $(-1,\infty)$ if n is odd, which is equivalent to $(-1)^n[\ln\nu_\alpha(x)]^{(n)}>0$ in $x\in(-1,\infty)$ for $n\in\mathbb{N}$ and $\alpha\ge1$. Hence, if $\alpha\ge1$, the function $\frac{[\Gamma(x+1)]^{1/x}}{(x+1)^\alpha}$ is logarithmically completely monotonic in $(-1,\infty)$.

Conversely, if the function $\frac{[\Gamma(x+1)]^{1/x}}{(x+1)^\alpha}$ is logarithmically completely monotonic in $(-1,\infty)$, then $[\ln\nu_\alpha(x)]'\le0$ which is equivalent to
$$\begin{aligned}\alpha&\ge\frac{x+1}{x^2}[x\psi(x+1)-\ln\Gamma(x+1)]\\&=\left(1+\frac{1}{x}\right)\left[\psi(x+1)-\frac{\ln\Gamma(x+1)}{x}\right]\\&\to\left(1+\frac{1}{x}\right)\left\{\ln(x+1)-\frac{1}{2(x+1)}-\frac{1}{12(x+1)^2}+O\left(\frac{1}{x+1}\right)\right.\\&\quad\left.-\frac{1}{x}\left[\left(x+\frac{1}{2}\right)\ln(x+1)-x-1+\frac{\ln(2\pi)}{2}+\frac{1}{12(x+1)}+O\left(\frac{1}{x+1}\right)\right]\right\}\\&\to1\end{aligned}$$
as $x\to\infty$ by using the following formulas (see [1, 29, 64, 65])
$$\ln\Gamma(x)=\left(x-\frac{1}{2}\right)\ln x-x+\frac{\ln(2\pi)}{2}+\frac{1}{12x}+O\left(\frac{1}{x}\right)\tag{33}$$
and
$$\psi(x)=\ln x-\frac{1}{2x}-\frac{1}{12x^2}+O\left(\frac{1}{x^2}\right)\tag{34}$$
as $x\to\infty$. □

Proof of Theorem 3. If $\alpha\le0$, the logarithmically complete monotonicity of the function $\frac{x^\alpha}{[\Gamma(x+1)]^{1/x}}$ in $(0,\infty)$ follows from the similar arguments as in the proofs of Theorem 2.

If $\frac{x^\alpha}{[\Gamma(x+1)]^{1/x}}$ in $(0,\infty)$ is logarithmically completely monotonic, then its logarithmic derivative

$$\frac{\alpha}{x}+\frac{\ln\Gamma(1+x)-x\psi(1+x)}{x^2}$$

is negative in $(0,\infty)$. Since

$$\lim_{x\to0^+}\frac{\ln\Gamma(1+x)-x\psi(1+x)}{x^2}=-\frac{\pi^2}{12} \tag{35}$$

by L'Hospital rule and

$$\lim_{x\to0^+}\frac{\alpha}{x}=\begin{cases}\infty, & \text{if } \alpha>0,\\ 0, & \text{if } \alpha=0,\\ -\infty, & \text{if } \alpha<0,\end{cases} \tag{36}$$

then it must hold that $\alpha\le 0$.

The rest proofs of Theorem 3 are similar to the proofs of Theorem 2, so we omit it. □

Proof of Theorem 4. Since $\tau(s,t)>0$, which has been proved in Theorem 2 by utilizing Bernoulli's inequality, it is clear that $\tau_0>0$. When $\alpha\le\frac{1}{1+\tau_0}<1$, from (32) it follows that $\mu'_{\alpha,n}(x)\le 0$ and $\mu_{\alpha,n}(x)$ is decreasing in $(-1,\infty)$ if n an even integer and that $\mu'_{\alpha,n}(x)\le0$ and $\mu_{\alpha,n}(x)$ is decreasing in $(-1,0)$ and $\mu'_{\alpha,n}(x)\ge0$ and $\mu_{\alpha,n}(x)$ is increasing in $(0,\infty)$ if n an odd integer. Since $\mu_{\alpha,n}(0)=0$ and $\lim_{x\to\infty}[\ln\nu_\alpha(x)]^{(n)}=0$, we have $[\ln\nu_\alpha(x)]^{(n)}<0$ for n being an even and $[\ln\nu_\alpha(x)]^{(n)}>0$ for n being an odd in $(-1,\infty)$, this implies that $(-1)^{n+1}[\ln\nu_\alpha(x)]^{(n)}>0$ in $(-1,\infty)$ for $n\in\mathbb{N}$. Therefore $\nu_\alpha(x)$ is strictly increasing and $(-1)^{n-1}\{[\ln\nu_\alpha(x)]'\}^{(n-1)}>0$ in $(-1,\infty)$ for $n\in\mathbb{N}$. Hence, if $\alpha\le\frac{1}{1+\tau_0}$, then the function $\frac{(x+1)^\alpha}{[\Gamma(x+1)]^{1/x}}$ is logarithmically completely monotonic in $(-1,\infty)$. The proof of Theorem 4 is complete. □

Proof of Theorem 5. This follows from modified arguments of above theorems. □

Proof of Theorem 6. Let $\mathcal{F}_\alpha(x)=\frac{f(x)}{f(x+\alpha)}$ for $\alpha>0$. Since $f(x)$ is logarithmically completely monotonic, by definition we have $(-1)^k[\ln f(x)]^{(k)}\ge0$ for $k\in\mathbb{N}$, which is equivalent to $[\ln f(x)]^{(2i)}\ge0$ and $[\ln f(x)]^{(2i-1)}\le0$ for $i\in\mathbb{N}$, and $[\ln f(x)]^{(2i)}$ is decreasing and $[\ln f(x)]^{(2i-1)}$ is increasing. So

$$[\ln\mathcal{F}_\alpha(x)]^{(2i)}=[\ln f(x)]^{(2i)}-[\ln f(x+\alpha)]^{(2i)}\ge0$$

and $[\ln\mathcal{F}_\alpha(x)]^{(2i-1)}\le0$ for $\alpha>0$ and $i\in\mathbb{N}$. The proof of Theorem 6 is complete. □

Proof of Theorem 7. Let

$$F_n(x)=\prod_{i=1}^n[f_i(x)]^{\alpha_i}.$$

Then

$$\ln F_n(x)=\sum_{i=1}^n\alpha_i\ln f_i(x)$$

and

$$(-1)^k[\ln F_n(x)]^{(k)}=\sum_{i=1}^n\alpha_i(-1)^k[\ln f_i(x)]^{(k)}$$

for $k\in\mathbb{N}$. Since $f_i(x)\in\mathcal{L}[I]$, that is, $(-1)^k[\ln f_i(x)]^{(k)}\ge0$, and $\alpha_i\ge0$, it is easy to see that $(-1)^k[\ln F_n(x)]^{(k)}\ge0$ for $k\in\mathbb{N}$. The proof of Theorem 7 is complete. □

Proof of Theorem 8. In [21, No. 0.430.1] the formula for the n-th derivative of a composite function is given by

$$\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\big[f\big(h(x)\big)\big]=\sum_{k=1}^{n}\frac{1}{k!}f^{(k)}\big(h(x)\big)U_k(x), \tag{37}$$

where

$$U_k(x)=\sum_{i=0}^{k-1}(-1)^i\binom{k}{i}[h(x)]^i\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}[h(x)]^{k-i}. \tag{38}$$

From this it is deduced that $-\frac{f'(h(x))}{f(h(x))}\in\mathcal{C}[I]$ since $-\frac{f'(x)}{f(x)}$ is a completely monotonic function, which is equivalent to $f(x)$ being logarithmically completely monotonic, and $h'(x)\in\mathcal{C}[I]$. Therefore, $(-1)^i\big[\frac{f'(h(x))}{f(h(x))}\big]^{(i)}\le 0$ on the interval I for nonnegative integer i.

Since $h'(x)\in\mathcal{C}[I]$, it is obtained that $(-1)^ih^{(i+1)}(x)\ge 0$ on the interval I for nonnegative integer i.

Hence, for $k\in\mathbb{N}$,

$$\begin{aligned}
(-1)^k\big[\ln f\big(h(x)\big)\big]^{(k)}&=(-1)^k\left[\frac{f'\big(h(x)\big)}{f\big(h(x)\big)}h'(x)\right]^{(k-1)}\\
&=(-1)^k\sum_{i=0}^{k-1}\binom{k-1}{i}\left[\frac{f'\big(h(x)\big)}{f\big(h(x)\big)}\right]^{(i)}h^{(k-i)}(x)\\
&=\sum_{i=0}^{k-1}\binom{k-1}{i}\left\{(-1)^i\left[\frac{f'\big(h(x)\big)}{f\big(h(x)\big)}\right]^{(i)}\right\}\big[(-1)^{k-i}h^{(k-i)}(x)\big]\ge 0.
\end{aligned}$$

The proof of Theorem 8 is complete. □

3. Remarks and applications of theorems

Remark 1. As said in [7] and done in various papers, the complete monotonicity for special functions has been established by proving the stronger statement that the function is logarithmically completely monotonic or is a Stieltjes transform. In some concrete cases it is often easier to establish that a function is logarithmically completely monotonic or is a Stieltjes transform than to verify directly the complete monotonicity. One of the important values of this paper might be owning to the standard or elementary proofs of some theorems in this paper.

Remark 2. It is remarked that many complete monotonicity results in [3, 4, 5, 18, 30, 63] and the references therein can be restated in terms of the logarithmically complete monotonicity indeed.

Remark 3. In [4] and [10, p. 83], the following result was given: Let f and g be functions such that $f\circ g$ is defined. If $f\in\mathcal{C}[(0,\infty)]$ and $g'\in\mathcal{C}[(0,\infty)]$, then $f\circ g\in\mathcal{C}[(0,\infty)]$. Since the exponential function $e^{-x}\in\mathcal{C}[(-\infty,\infty)]$, hence $\mathcal{L}[(0,\infty)]\subset\mathcal{C}[(0,\infty)]$, a logarithmically completely monotonic function is also completely monotonic. This gives an alternative proof of the conclusion $\mathcal{L}[(0,\infty)]\subset\mathcal{C}[(0,\infty)]$.

Remark 4. In [5, 7] it is shown that $\frac{[\Gamma(1+x)]^{1/x}}{x}\in\mathcal{S}$ and $\frac{1}{[\Gamma(1+x)]^{1/x}}\in\mathcal{S}$. Although these results are stronger than Theorem 1 and parts of Theorem 3, but the ranges of x are extended to $(-1,\infty)$ in Theorem 1 and a parameter α is considered in Theorem 3. Consequently, Theorem 1 and Theorem 3 still make sense.

Remark 5. It is noted that a non-elementary argument for the sufficient part of Theorem 2 was provided by an anonymous referee of this paper as follows. Looking at what is really written in [5], which builds on a technique from [9], it is easy to obtain that

$$h(z) = \frac{\ln\Gamma(z+1)}{z} - \alpha\ln(z+1) = c + \int_1^\infty\left(\frac{1}{t+z} - \frac{t}{1+t^2}\right)[\alpha - M(t)]\,\mathrm{d}\,t$$

with

$$c = -\gamma + \sum_{k=1}^\infty\left(\frac{1}{k} - \arctan\frac{1}{k}\right)$$

and $M(t) = \frac{k-1}{t}$ for $t\in(k-1,k]$ and $k=2,3,\dots$. For $\alpha\ge1$ one has $\alpha\ge M(t)$. Accordingly,

$$h'(t) = -\int_1^\infty\frac{\alpha-M(t)}{(t+z)^2}\,\mathrm{d}\,t<0, \qquad z>-1, \tag{39}$$

that is, the function h is decreasing with $-h'\in\mathcal{C}[(-1,\infty)]$, which is the sufficient part of Theorem 2.

Remark 6. Now we give a weaker proof of Theorem 2 and Theorem 4 by another approach.

It is well-known (see [1, 64, 65]) that for $x>0$ and $r>0$

$$\frac{1}{x^r} = \frac{1}{\Gamma(r)}\int_0^\infty t^{r-1}e^{-xt}\,\mathrm{d}\,t. \tag{40}$$

Substituting (40) and (23) into the second line of formula (32) yields

$$\begin{aligned}
\mu'_{\alpha,n}(x) &= x^n\left[\psi^{(n)}(x+1) + \frac{(-1)^n(n-1)!\alpha}{(x+1)^n} + \frac{(-1)^n n!\alpha}{(x+1)^{n+1}}\right]\\
&= x^n\bigg\{(-1)^{n+1}\int_0^\infty\frac{t^ne^{-(x+1)t}}{1-e^{-1}}\,\mathrm{d}\,t\\
&\quad+(-1)^n\alpha\bigg[\int_0^\infty t^{n-1}e^{-(x+1)t}\,\mathrm{d}\,t+\int_0^\infty t^ne^{-(x+1)t}\,\mathrm{d}\,t\bigg]\bigg\}\\
&= (-1)^nx^n\int_0^\infty\left[\alpha-\frac{et}{(e-1)(1+t)}\right](1+t)t^{n-1}e^{-(x+1)t}\,\mathrm{d}\,t.
\end{aligned}$$

It is clear that the function $\frac{t}{1+t}$ is increasing in $[0,\infty)$ with $0\le\frac{t}{1+t}<1$. Thus, the function $(-1)^n\frac{\mu'_{\alpha,n}(x)}{x^n}$ is non-positive for $\alpha\le0$ and positive for $\alpha\ge\frac{e}{e-1}$ in $(-1,\infty)$. Then

(1). For n is even,

$$\mu'_{\alpha,n}(x)\begin{cases}\le0 & \text{if } \alpha\le0,\\ >0 & \text{if } \alpha\ge\frac{e}{e-1}.\end{cases} \tag{41}$$

and

$$\mu_{\alpha,n}(x)\begin{cases}\text{is decreasing if } \alpha\le0,\\ \text{is increasing if } \alpha\ge\frac{e}{e-1},\end{cases} \tag{42}$$

hence, from $\mu_{\alpha,n}(0)=0$, it follows that

$$\begin{cases}\text{if } \alpha\le0, & \mu_{\alpha,n}(x)\begin{cases}>0, & x\in(-1,0),\\ <0, & x\in(0,\infty),\end{cases}\\ \text{if } \alpha\ge\dfrac{e}{e-1}, & \mu_{\alpha,n}(x)\begin{cases}<0, & x\in(-1,0),\\ >0, & x\in(0,\infty),\end{cases}\end{cases} \tag{43}$$

therefore,

$$[\ln \nu_\alpha(x)]^{(n)} = \frac{\mu_{\alpha,n}(x)}{x^{n+1}} \begin{cases} \le 0 & \text{if } \alpha \le 0, \\ \ge 0 & \text{if } \alpha \ge \frac{e}{e-1}; \end{cases} \tag{44}$$

(2). For n is odd,

$$\frac{\mu'_{\alpha,n}(x)}{x^n} \begin{cases} \ge 0 & \text{if } \alpha \le 0, \\ < 0 & \text{if } \alpha \ge \frac{e}{e-1}, \end{cases}$$

consequently,

$$\begin{cases} \text{if } \alpha \le 0, & \mu'_{\alpha,n}(x) \begin{cases} \ge 0 & x \in (0,\infty), \\ \le 0 & x \in (-1,0), \end{cases} \\ \text{if } \alpha \ge \dfrac{e}{e-1}, & \mu'_{\alpha,n}(x) \begin{cases} < 0 & x \in (0,\infty), \\ > 0 & x \in (-1,0), \end{cases} \end{cases}$$

and

$$\begin{cases} \text{if } \alpha \le 0, & \mu_{\alpha,n}(x) \begin{cases} \text{is increasing in } (0,\infty), \\ \text{is decreasing in } (-1,0), \end{cases} \\ \text{if } \alpha \ge \dfrac{e}{e-1}, & \mu_{\alpha,n}(x) \begin{cases} \text{is decreasing in } (0,\infty), \\ \text{is increasing in } (-1,0), \end{cases} \end{cases}$$

as a result, from $\mu_{\alpha,n}(0) = 0$, it is easy to obtain that

$$\mu_{\alpha,n}(x) \begin{cases} \ge 0 & \text{for } \alpha \le 0, \\ \le 0 & \text{for } \alpha \ge \frac{e}{e-1}, \end{cases}$$

which is equivalent to

$$[\ln \nu_\alpha(x)]^{(n)} = \frac{\mu_{\alpha,n}(x)}{x^{n+1}} \begin{cases} \ge 0 & \text{for } \alpha \le 0, \\ \le 0 & \text{for } \alpha \ge \frac{e}{e-1}. \end{cases}$$

In conclusion, we have

$$(-1)^n [\ln \nu_\alpha(x)]^{(n)} \begin{cases} \le 0 & \text{if } \alpha \le 0, \\ \ge 0 & \text{if } \alpha \ge \frac{e}{e-1}. \end{cases} \tag{45}$$

Remark 7. It has been proved in the proof of Theorem 2 that $\tau(s,t) > 0$ for $(s,t) \in (0,\infty) \times (0,\infty)$. Now we give an upper bound of the function $\tau(s,t)$ on $(0,\infty) \times (0,\infty)$.

Let $s = \mu t$ for $\mu \in (0,\infty)$. Then we have

$$\tau(\mu t, t) = \frac{1}{\mu}\left[1 - \frac{(\mu+1)t+1}{1+t}\left(\frac{t}{1+t}\right)^{\mu t}\right]. \tag{46}$$

Since the function $\frac{(\mu+1)t+1}{1+t}$ is strictly increasing with $t \in (0,\infty)$ for fixed $\mu \in (0,\infty)$, it follows that

$$1 < \frac{(\mu+1)t+1}{1+t} < \mu + 1. \tag{47}$$

Since the function $\left(1 + \frac{1}{t}\right)^t$ is strictly increasing, we see that

$$\left(\frac{t}{1+t}\right)^{\mu t} = \left[\frac{1}{(1+1/t)^t}\right]^\mu \tag{48}$$

is strictly decreasing with $t \in (0,\infty)$ for fixed $\mu \in (0,\infty)$, therefore

$$\frac{1}{e^\mu} < \left(\frac{t}{1+t}\right)^{\mu t} < 1. \tag{49}$$

Combining (46), (47) and (49) produces

$$\tau(\mu t, t) < \frac{1}{\mu}\left(1 - \frac{1}{e^\mu}\right) < \lim_{\mu\to 0}\left[\frac{1}{\mu}\left(1 - \frac{1}{e^\mu}\right)\right] = 1. \tag{50}$$

Since $\mu \in (0,\infty)$ and $t \in (0,\infty)$ are arbitrary, so we have $\tau(s,t) < 1$ for $(s,t) \in (0,\infty)\times(0,\infty)$.

Recently, the upper bound of $\tau(s,t)$ was improved from 1 to $\frac{1}{3}$ in [57] and further to $\frac{3}{10}$ in [41].

Remark 8. By definition, it is clear that one of the necessary conditions such that $\frac{(x+1)^\alpha}{[\Gamma(x+1)]^{1/x}} \in \mathcal{L}[(-1,\infty)]$ is $[\ln \nu_\alpha(x)]' \ge 0$ in $(-1,\infty)$, where $\nu_\alpha(x)$ is defined by (29), which is equivalent to

$$\alpha \le \frac{(x+1)[x\psi(x+1) - \ln\Gamma(x+1)]}{x^2}.$$

Combining this with Theorem 4 yields

$$\tau_0 \ge \frac{x^2}{(x+1)[x\psi(x+1) - \ln\Gamma(x+1)]} - 1 \tag{51}$$

for $x \in (-1,\infty)$.

Straightforward numerical computation by the software Mathematica shows that the maximum τ_2 of $\tau(2,t)$ in $(0,\infty)$ is

$$\tau\left(2, \frac{2+\sqrt{7}}{3}\right) = \frac{1}{2}\left[\frac{2+\sqrt{7}}{3} - \frac{\left(2+\sqrt{7}\right)^3\left(3+\frac{2+\sqrt{7}}{3}\right)}{27\left(1+\frac{2+\sqrt{7}}{3}\right)^3}\right] = 0.264076\cdots \tag{52}$$

and the maximum τ_3 of $\tau(3,t)$ in $(0,\infty)$ is

$$\tau\left(3, \frac{5}{9} + \frac{\sqrt[3]{2836 - 54\sqrt{406}}}{18} + \frac{\sqrt[3]{1418 + 27\sqrt{406}}}{9\sqrt[3]{4}}\right) = 0.271807\cdots. \tag{53}$$

If $\alpha \le \frac{1}{1+\tau_2} = 0.79\cdots$, then $\mu'_{\alpha,2}(x) \le 0$ and $\mu_{\alpha,2}(x)$ decreases in $(-1,\infty)$. Since $\mu_{\alpha,2}(0) = 0$ and $\lim_{x\to\infty}[\ln \nu_\alpha(x)]^{(2)} = 0$, it is obtained that $[\ln \nu_\alpha(x)]^{(2)} < 0$. Therefore the function $\nu_\alpha(x) = \frac{[\Gamma(x+1)]^{1/x}}{(x+1)^\alpha}$ is strictly increasing and logarithmically concave for $\alpha \le \frac{1}{1+\tau_2}$ in $(-1,\infty)$. If $\alpha \le \frac{1}{1+\tau_3} = 0.78\cdots$, then $\mu'_{\alpha,3}(x) < 0$ and $\mu_{\alpha,3}(x)$ decreases in $(-1,0)$ and $\mu'_{\alpha,3}(x) > 0$ and $\mu_{\alpha,3}(x)$ increases in $(0,\infty)$. Thus $\mu_{\alpha,3}(x) \ge 0$ and then $[\ln \nu_\alpha(x)]^{(3)} > 0$ in $(-1,\infty)$. Hence $[\ln \nu_\alpha(x)]^{(2)}$ is strictly increasing in $(-1,\infty)$ if $\alpha \le \frac{1}{1+\tau_3}$.

Remark 9. It is proved in [42, 43] that

$$\frac{\ln\Gamma(x+1)}{x} - \ln x + 1 = \ln\frac{[\Gamma(x+1)]^{1/x}}{x} + 1 \in \mathcal{C}[(0,\infty)]$$

and tends to ∞ as $x \to 0$ and to 0 as $x \to \infty$. A similar result was found in [63]: The function

$$1 + \frac{\ln\Gamma(x+1)}{x} - \ln(x+1) = \ln\frac{[\Gamma(x+1)]^{1/x}}{x+1} + 1$$

belongs to $\mathcal{C}[(-1,\infty)]$ and tends to 1 as $x \to -1$ and to 0 as $x \to \infty$. These are special cases of our main results, for examples, Theorem 2 and Theorem 3.

In what follows, as applications of our main results, we would like to deduce some consequences of the theorems stated in Section 1.

Proposition 1. *The function*

$$\frac{[\Gamma(x+\alpha+1)]^{1/(x+\alpha)}}{[\Gamma(x+1)]^{1/x}} \tag{54}$$

belongs to $\mathcal{L}[(-1,\infty)]$ *if and only if* $\alpha>0$.

For $\alpha\ge 1$ *and* $\beta>0$*, the function*

$$\frac{[\Gamma(x+1)]^{1/x}}{[\Gamma(x+1+\beta)]^{1/(x+\beta)}}\left(1+\frac{\beta}{x+1}\right)^{\alpha} \tag{55}$$

belongs to $\mathcal{L}[(-1,\infty)]$. *For* $\beta>0$ *and any given real number* α *satisfying* $\alpha\le\frac{1}{1+\tau_0}<1$*, the reciprocal of the function defined by* (55) *for* $\beta>0$ *belongs to* $\mathcal{L}[(-1,\infty)]$.

For $\alpha\ge 1$ *and* $\beta>0$*, the function*

$$\frac{[\Gamma(x+1)]^{1/x}}{[\Gamma(x+1+\beta)]^{1/(x+\beta)}}\left(1+\frac{\beta}{x}\right)^{\alpha} \tag{56}$$

belongs to $\mathcal{L}[(0,\infty)]$. *For* $\alpha\le 0$ *and* $\beta>0$*, the reciprocal of the function defined by* (56) *belongs to* $\mathcal{L}[(0,\infty)]$.

Proof. These follow from combining Theorem 6 with Theorem 1, Theorem 2, Theorem 3, and Theorem 4. □

Remark 10. In [26, 31], among other things, the following monotonicity results were obtained:

$$[\Gamma(1+k)]^{1/k}<[\Gamma(2+k)]^{1/(k+1)}, \quad k\in\mathbb{N};$$
$$\left[\Gamma\left(1+\frac{1}{x}\right)\right]^{x} \text{ decreases with } x>0.$$

These are extended and generalized in [36, 37, 38, 55], among other things: The function $[\Gamma(r)]^{1/(r-1)}$ is increasing in $r>0$. Clearly, Theorem 1 generalizes these results and extends them for the range of the argument.

The first conclusion in Proposition 1 shows that the sequences

$$\frac{\sqrt[k]{k!}}{\sqrt[m+k]{(m+k)!}} \quad\text{and}\quad \frac{\left[\sqrt[k]{k!}\right]\left[\sqrt[k+m+n]{(k+m+n)!}\right]}{\left[\sqrt[k+m]{(k+m)!}\right]\left[\sqrt[k+n]{(k+n)!}\right]} \tag{57}$$

are increasing with $k\in\mathbb{N}$ for given natural numbers m and n.

Remark 11. The results in Proposition 1 generalize and extend those of [48, 49].

Define

$$Q_{a,b}(x)=\frac{[\Gamma(x+a+1)]^{1/(x+a)}}{[\Gamma(x+b+1)]^{1/(x+b)}} \tag{58}$$

for nonnegative real numbers a and b. J. Sándor [59] established that $Q_{1,0}$ is decreasing on $(1,\infty)$. In [5] Alzer and Berg proved that $[Q_{a,b}(x)]^c$ is completely monotonic with $x\in(0,\infty)$ if and only if $a\ge b$ for $c>0$. The following proposition extends the ranges of variables a,b and x in [5] and can be regarded as a generalization of Proposition 1 above.

Proposition 2. *Let* $a,b\in\mathbb{R}$ *and* $c>0$. *Then* $[Q_{a,b}(x)]^c\in\mathcal{L}[(-(1+b),\infty)]$ *if and only if* $a>b$.

Proof. From Theorem 1, it is clear that

$$\frac{1}{[\Gamma(x+a+1)]^{1/(x+a)}}\in\mathcal{L}[(-(1+a),\infty)]$$

for $a \in \mathbb{R}$. From Theorem 6 it follows that the function $Q_{a,b}(x)$ is logarithmically completely monotonic in $(-(1+a),\infty)\cap(-(1+b),\infty)=(-(1+b),\infty)$ for $a>b$. So does the function $[Q_{a,b}(x)]^c$ for $c>0$.

If $[Q_{a,b}(x)]^c$ is logarithmically completely monotonic for $c>0$, then the derivative $\{[\ln Q_{a,b}(x)]^c\}'=c[g'(x+a)-g'(x+b)]<0$, where $g(x)$ is defined by (24) and $g'(x)$ is strictly decreasing in $(-1,\infty)$, since $g''(x)=\int_0^1 t^2\psi''(xt+1)\,\mathrm{d}\,t<0$ from (28) and (22). Therefore, there must be $a>b$. □

Proposition 3. *Let f be a logarithmically completely monotonic function and g a completely monotonic function. Then the function*

$$f\biggl(a+b\int_\alpha^x g(t)\,\mathrm{d}\,t\biggr)$$

is logarithmically completely monotonic on an interval I if it is defined on I, where b is positive and $\alpha\in I$.

In particular, if f is logarithmically completely monotonic, then the following functions are also logarithmically completely monotonic:

$$f(ax^\alpha+b),\quad \text{where } a \text{ is nonnegative numbers and } 0\le\alpha\le1, \tag{59}$$

$$f\bigl(a+b\ln(1+x)\bigr),\quad \text{where } b \text{ is nonnegative}, \tag{60}$$

$$f(1-e^{-x}), \tag{61}$$

$$f\bigl(\arctan\sqrt{x}\,\bigr). \tag{62}$$

If $f(x)$ is completely monotonic on an interval I, then the function $[A-f(x)]^{-\mu}$ is logarithmically completely monotonic on I, where $A>f(x)$ for $x\in I$ and $\mu\ge0$.

Proof. These are direct consequences of Theorem 8. □

Remark 12. The following are also logarithmically completely monotonic functions:

$$\exp(-ax^\alpha),\quad \text{where } a\ge0 \text{ and } 0\le\alpha\le1, \tag{63}$$

$$[a+b\ln(1+x)]^{-\mu},\quad \text{where } a\ge0,\ b\ge0 \text{ and } \mu>0, \tag{64}$$

$$(a-be^{-x})^{-\mu},\quad \text{where } a\ge b>0 \text{ and } \mu\ge0. \tag{65}$$

Remark 13. Finally, we pose an open problem: Let $\tau_0=\tau(s_0,t_0)$ be the maximum value of $\tau(s,t)$ defined by (17) on the set $\mathbb{N}\times(0,\infty)$. Then $\frac{(x+1)^\alpha}{[\Gamma(x+1)]^{1/x}}\in\mathcal{L}[(-1,\infty)]$ if and only if $\alpha\le\frac{1}{1+\tau_0}<1$.

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